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BENDING SOLUTION OF HIGH-ORDER REFINED SHEAR DEFORMATION THEORY FOR RECTANGULAR COMPOSITE PLATES

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Abstract—A new high-order refined shear deformation theory based on Reissner's mixed variational principle in conjunction with the state–space concept is used to determine the deflections and stresses for rectangular cross-ply composite plates. A zig-zag shaped function and Legendre polynomials are introduced to approximate the in-plane displacement distributions across the plate thickness. Numerical results are presented with different edge conditions, aspect ratios, lamination schemes and loadings. A comparison with the exact solutions obtained by Pagano and the results by Khdeir indicates that the present theory accurately estimates the in-plane responses.

1. INTRODUCTION

Three-dimensional elasticity solutions for the bending of simply supported thick orthotropic rectangular plates and laminates were obtained by Srinivas and Rao (1970), Srinivas *et al.* (1970), Hussainy and Srinivas (1975) and Pagano (1970). The Navier solution of simply supported rectangular plates was developed by Whitney and Leissa (1969) for classical laminate theory, Whitney and Pagano (1970), Bert and Chen (1978) and Reddy and Chao (1981) for the first-order shear deformation (i.e. the Reissner–Mindlin plate) theory, and by Reddy (1984a,b) and Reddy and Phan (1985) for refined shear deformation theories. The Lévy type solutions were developed by Reddy *et al.* (1987) for symmetric laminates with different combinations of free, clamped and simply supported boundary conditions by using the first-order shear deformation theory. Khdeir *et al.* (1987) later extended Reddy's work by using refined shear deformation theory.

Murakami (1986) proposed an improved in-plane response theory based on Reissner's (1984) mixed variational principle and applied it to cylindrical bending problem of laminated plates, the improvement was achieved by including a zig-zag shaped function to approximate the in-plane displacements across the thickness. However, this theory cannot exactly describe the deformation of the anti-symmetric and irregular laminated plates.

Based upon Murakami's theory, Legendre polynomials are introduced in the displacement field and the transverse normal strain is also included in present theory so that the in-plane displacement distribution for arbitrary laminated configurations can be determined exactly. The advantage of using Reissner's mixed variation principle is that it automatically yields the appropriate shear correction factors for the transverse shear constitutive equations. Other attractive features of the present theory are: (1) the continuity condition of transverse shear stresses at the interfaces is satisfied; (2) the effects of the transverse shear and transverse normal strains are accounted; (3) the number of equations remains unchanged as the number of layers increases.

The accuracy of the present theory is examined by applying it to bending problem of rectangular laminates with two opposite edges simply supported and the remaining edges subject to a combination of free, simply supported and clamped boundary conditions. Different aspect ratios, lamination schemes and loadings are considered. The state–space concept is used to solve the ordinary differential equations.

2. GOVERNING EQUATIONS

Consider an N-layer laminated composite plate, shown in Fig. 1. The following notation, $(\ )^{(k)}$, $k = 1, 2 \ldots N$, will designate quantities associated with the kth layer. The thickness of each layer is $n^{(k)}h$. Unless otherwise specified, the usual Cartesian indicial notation is employed where Latin and Greek indices range from 1 to 3 and 1 to 2, respectively. Repeated indices imply the summation convection and $(\ )_{,i}$ denotes partial differentiation with respect to x_i.

Constitutive equations for orthotropic layers (Murakami, 1986):

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}^{(k)} = \begin{bmatrix} \bar{c}_{11} & \bar{c}_{12} & 0 \\ \bar{c}_{12} & \bar{c}_{22} & 0 \\ 0 & 0 & c_{66} \end{bmatrix}^{(k)} \begin{bmatrix} e_{11} \\ e_{22} \\ 2e_{12} \end{bmatrix}^{(k)} + \begin{bmatrix} \dfrac{c_{13}}{c_{33}} \\ \dfrac{c_{23}}{c_{33}} \\ 0 \end{bmatrix}^{(k)} \sigma_{33}^{(k)} \tag{1a}$$

$$\begin{bmatrix} e_{33} \\ 2e_{23} \\ 2e_{31} \end{bmatrix}^{(k)} = -\begin{bmatrix} \dfrac{c_{13}}{c_{33}} & \dfrac{c_{23}}{c_{33}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{(k)} \begin{bmatrix} e_{11} \\ e_{22} \\ 2e_{12} \end{bmatrix}^{(k)} + \begin{bmatrix} \dfrac{1}{c_{33}} & 0 & 0 \\ 0 & \dfrac{1}{c_{44}} & 0 \\ 0 & 0 & \dfrac{1}{c_{55}} \end{bmatrix}^{(k)} \begin{bmatrix} \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}^{(k)}; \tag{1b}$$

strain–displacement relations

$$e_{ij}^{(k)} = \tfrac{1}{2}(u_{i,j}^{(k)} + u_{j,i}^{(k)}); \tag{2}$$

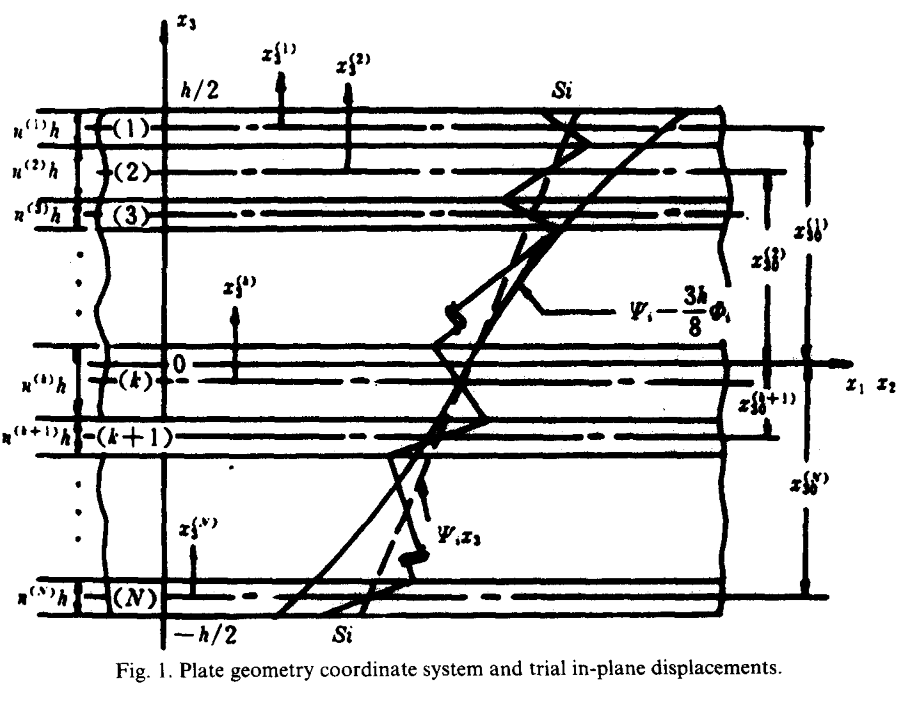

Fig. 1. Plate geometry coordinate system and trial in-plane displacements.

interface continuity conditions

$$u_i^{(k)} = u_i^{(k+1)} \quad \text{and} \quad \sigma_{3i}^{(k)} = \sigma_{3i}^{(k+1)} \quad k = 1,2\ldots N-1; \tag{3}$$

upper and lower surface stress conditions

$$\sigma_{3i}^{(1)} = T_i^+ \quad \text{on} \quad x_3 = \frac{h}{2} \tag{4a}$$

$$\sigma_{3i}^{(N)} = T_i^- \quad \text{on} \quad x_3 = -\frac{h}{2}. \tag{4b}$$

Reissner's mixed variational principle was applied to N-layer composite plate whose middle surface occupies a domain D in the x_1, x_2-plane:

$$\iint_D \left[\sum_k \int_{A^{(k)}} \{\delta e_{ij}^{(k)}\sigma_{ij}^{(k)} + [u_{\alpha,3}^{(k)} + u_{3,\alpha}^{(k)} - 2e_{3\alpha}^{(k)}(\ldots)]\delta\tau_{3\alpha}^{(k)} + [u_{3,3}^{(k)} - e_{33}^{(k)}(\ldots)]\delta\tau_{33}^{(k)}\}\,\mathrm{d}x_3\right]\mathrm{d}x_1\,\mathrm{d}x_2$$
$$= \int_{\partial D_T}\left[\sum_k \int_{A^{(k)}} \delta u_i^{(k)v} T_i^{(k)}\mathrm{d}x_3\right]\mathrm{d}s + \iint_D \left[\delta u_i^{(1)}\left(x_1, x_2, \frac{h}{2}\right)T_i^+ - \delta u_i^{(N)}\left(x_1, x_2, -\frac{h}{2}\right)T_i^-\right]\mathrm{d}x_1\,\mathrm{d}x_2, \tag{5}$$

where ∂D_T denotes the boundary of D with outward normal v_α on which tractions T_i are specified and $A^{(k)}$ represents the x_3-domain occupied by the kth layer. Also $\tau_{3i}^{(k)}$ denote the approximate transverse stresses and $e_{3i}^{(k)}\ldots$ implies the appropriate right-hand side of eqn (1b).

The high-order laminated plate theory, which takes into account the effect of transverse shear strains, is obtained by including the Legendre polynomials of order $n = 1, 2, 3$, with respect to the x_3-coordinate to a zig-zag in-plane displacement variation of amplitude $S_i(x_1, x_2)$ across the plate thickness.

The appropriate trial functions used in connection with Reissner's mixed variational principle eqn (5) are taken to be:
(a) trial displacement field (see Fig. 1)

$$u_i^{(k)}(x_1, x_2, x_3) = U_i(x_1, x_2) + \frac{h}{2}\Psi_i(x_1, x_2)P_1(\zeta) + S_i(x_1, x_2)(-1)^k\frac{2}{n^{(k)}h}x_3^{(k)}$$
$$+ \left(\frac{h}{2}\right)^2 \xi_i(x_1, x_2)P_2(\zeta) + \left(\frac{h}{2}\right)^3 \Phi_i(x_1, x_2)P_3(\zeta), \tag{6}$$

where $\zeta \equiv 2x_3/h$ and $P_n(\zeta)$ are the Legendre polynomials of order n and $\Phi_3 \equiv 0$, $x_3^{(k)}$ is a local x_3-coordinate system with its origin at the center $x_{30}^{(k)}$ of the kth layer, i.e.

$$x_3^{(k)} \equiv x_3 - x_{30}^{(k)}; \tag{7}$$

(b) trial transverse and normal stresses

$$\tau_{3\alpha}^{(k)}(x_1, x_2, x_3) = Q_\alpha^{(k)}(x_1, x_2)F_1(z) + R_\alpha^{(k)}(x_1, x_2)F_2(z)$$
$$+ J_\alpha^{(k)}(x_1, x_2)F_3(z) + [T_\alpha^{(k-1)}(x_1, x_2) + T_\alpha^{(k)}(x_1, x_2)]F_4(z)$$
$$+ [T_\alpha^{(k-1)}(x_1, x_2) - T_\alpha^{(k)}(x_1, x_2)]F_5(z) \tag{8a}$$

$$\tau_{33}^{(k)}(x_1,x_2,x_3) = Q_3^{(k)}(x_1,x_2)F_1(z)+R_3^{(k)}(x_1,x_2)F_6(z) + J_3^{(k)}(x_1,x_2)F_3(z)+I_3^{(k)}(x_1,x_2)F_7(z) + [T_3^{(k-1)}(x_1,x_2)+T_3^{(k)}(x_1,x_2)]F_4(z) + [T_3^{(k-1)}(x_1,x_2)-T_3^{(k)}(x_1,x_2)]F_8(z), \tag{8b}$$

where

$$F_1(z) = \frac{5}{n^{(k)}h}\left(21z^4-\frac{15}{2}z^2+\frac{9}{16}\right), \quad F_2(z) = \frac{-30}{(n^k h)^2}(4z^3-z)$$

$$F_3(z) = \frac{-105}{(n^{(k)}h)^3}\left(20z^4-6z^2+\frac{1}{4}\right), \quad F_4(z) = 35z^4-\frac{15}{2}z^2+\frac{3}{16}$$

$$F_5(z) = 10z^3-\frac{3}{2}z, \quad F_6(z) = \frac{105}{(n^k h)^2}\left(36z^5-14z^3+\frac{5}{4}z\right)$$

$$F_7(z) = \frac{-315}{(n^{(k)}h)^4}(112z^5-40z^3+3z), \quad F_8(z) = 126z^5-35z^3+\frac{15}{8}z \tag{9}$$

$$z \equiv \frac{x_3^{(k)}}{n^{(k)}h} \quad -\frac{1}{2} \leqslant z \leqslant \frac{1}{2}.$$

Also

$$(Q_i^{(k)}, R_i^{(k)}, J_i^{(k)}) \equiv \int_{A^{(k)}} (1, x_3^{(k)}, x_3^{(k)2})\tau_{3i}^{(k)}\,\mathrm{d}x_3 \tag{10a}$$

$$I_3^{(k)} \equiv \int_{A^{(k)}} x_3^{(k)3}\tau_{33}^{(k)}\,\mathrm{d}x_3. \tag{10b}$$

In eqn (8), $T_i^{(k-1)}$ and $T^{(i)}$ are the values of $\tau_{3i}^{(k)}$ at the top and bottom surfaces of the kth layer, respectively, from eqn (4)

$$T_i^{(0)} = T_i^+ \quad \text{and} \quad T_i^{(N)} = T_i^-. \tag{11}$$

The functions $F_i(z)$, $i = 1\ldots 8$ are obtained by first noting that eqn (6) yields cubic variations across the plate thickness of in-plane stresses. From the equilibrium equations (i.e. $\sigma_{ji,j}^{(k)} = 0$), transverse stresses $\tau_{3\alpha}^{(k)}$ and $\tau_{33}^{(k)}$ may, as a result, be represented by polynomials of degree 4 and 5 in z, respectively. Their corresponding coefficients are then computed by using eqns (10a, b). This yields the functions $F_i(z)$.

Substituting eqns (6) and (8) into eqn (5), using Gauss' theorem and the orthogonality relationship of the Legendre polynomials, one obtains laminated plate equations:
(a) equilibrium equations

$$N_{\alpha i,\alpha}+T_i^+-T_i^- = 0 \tag{12a}$$

$$M_{\alpha i,\alpha}-N_{3i}+\frac{h}{2}(T_i^++T_i^-) = 0 \tag{12b}$$

$$Z_{\alpha i,\alpha}-K_{3i}-[T_i^+-(-1)^N T_i^-] = 0 \tag{12c}$$

$$L_{\alpha i,\alpha}-3M_{3i}+\frac{h^2}{4}(T_i^+-T_i^-) = 0 \tag{12d}$$

$$P_{\beta\alpha,\beta}-\left(5L_{3\alpha}+\frac{h^2}{4}N_{3\alpha}\right)+\frac{h^3}{8}(T_\alpha^+ + T_\alpha^-)=0, \tag{12e}$$

$$[N_{\alpha\beta}, M_{\alpha\beta}, Z_{\alpha\beta}, L_{\alpha\beta}, P_{\alpha\beta}] \equiv \sum_{k=1}^{N}\int_{A^{(k)}}\left[1, \frac{h}{2}P_1(\zeta), (-1)^k\frac{2x_3^{(k)}}{n^{(k)}h}, \left(\frac{h}{2}\right)^2 P_2(\zeta), \left(\frac{h}{2}\right)^3 P_3(\zeta)\right]\sigma_{\alpha\beta}^{(k)}\,dx_3 \tag{13a}$$

$$[N_{3i}, M_{3i}, K_{3i}, Z_{3i}, L_{3i}] \equiv \sum_{k=1}^{N}\int_{A^{(k)}}\left[1, \frac{h}{2}P_1(\zeta), (-1)^k\frac{2}{n^{(k)}h}, (-1)^k\frac{2}{n^{(k)}h}x_3^{(k)}, \left(\frac{h}{2}\right)^2 P_2(\zeta)\right]\tau_{3i}^{(k)}\,dx_3\,; \tag{13b}$$

(b) constitutive equations

(1) for transverse stresses

$$Q_\alpha^{(k)}-\frac{8J_\alpha^{(k)}}{(n^{(k)}h)^2}+\frac{n^{(k)}h}{30}(T_\alpha^{(k-1)}+T_\alpha^{(k)}) = \frac{2}{5}hn^{(k)}\tilde{c}_\alpha^{(k)}\left[U_{3,\alpha}+\Psi_\alpha+S_\alpha(-1)^k\frac{2}{n^{(k)}h}+hn_0^{(k)}(\Psi_{3,\alpha}+3\xi_\alpha) + \frac{h^2}{2}\left(3n_0^{(k)2}-\frac{1}{4}\right)\xi_{3,\alpha}+\frac{3h^2}{2}\left(5n_0^{(k)2}-\frac{1}{4}\right)\Phi_\alpha\right] \tag{14a}$$

$$\frac{1}{h}R_\alpha^{(k)}-\frac{n^{(k)2}h}{40}(T_\alpha^{(k-1)}-T_\alpha^{(k)}) = \frac{7h^2}{120}n^{(k)3}\tilde{c}_\alpha^{(k)}\left[\Psi_{3,\alpha}+3\xi_\alpha+S_{3,\alpha}(-1)^k\frac{2}{n^{(k)}h}+3hn_0^{(k)}(\xi_{3,\alpha}+5\Phi_\alpha)\right] \tag{14b}$$

$$Q_\alpha^{(k)}-\frac{14J_\alpha^{(k)}}{(n^{(k)}h)^2}+\frac{n^{(k)}h}{12}(T_\alpha^{(k-1)}+T_\alpha^{(k)}) = -\frac{3h^3}{40}n^{(k)3}\tilde{c}_\alpha^{(k)}(\xi_{3,\alpha}+5\Phi_\alpha) \tag{14c}$$

$$-\frac{1}{\tilde{c}_\alpha^{(k)}}\left[\frac{1}{12}Q_\alpha^{(k)}-\frac{5J_\alpha^{(k)}}{3(n^{(k)}h)^2}+\frac{3R_\alpha^{(k)}}{7n^{(k)}h}\right]-\frac{1}{\tilde{c}_\alpha^{(k+1)}}\left[\frac{1}{12}Q_\alpha^{(k+1)}-\frac{5J_\alpha^{(k+1)}}{3(n^{(k+1)}h)^2}-\frac{3R_\alpha^{(k+1)}}{7n^{(k+1)}h}\right] = \frac{h}{126}\left[\frac{-n^{(k)}}{\tilde{c}_\alpha^{(k)}}T_\alpha^{(k-1)}+8\left(\frac{n^{(k)}}{\tilde{c}_\alpha^{(k)}}+\frac{n^{(k+1)}}{\tilde{c}_\alpha^{(k+1)}}\right)T_\alpha^{(k)}-\frac{n^{(k+1)}}{\tilde{c}_\alpha^{(k+1)}}T_\alpha^{(k+1)}\right]; \tag{14d}$$

(2) for normal stresses

$$Q_3^{(k)}-\frac{8J_3^{(k)}}{(n^{(k)}h)^2}+\frac{n^{(k)}h}{30}(T_3^{(k-1)}+T_3^{(k)}) = \frac{2h}{5}n^{(k)}c_{33}^{(k)}\left[\Psi_3+S_3(-1)^k\frac{2}{n^{(k)}h}+3hn_0^{(k)}\xi_3\right]+\frac{2h}{5}n^{(k)}\left[\tilde{U}+hn_0^{(k)}\tilde{\Psi} + \frac{h^2}{2}\left(3n_0^{(k)2}-\frac{1}{4}\right)\tilde{\xi}+\frac{h^3}{2}\left(5n_0^{(k)3}-\frac{3}{4}n_0^{(k)}\right)\tilde{\Phi}\right], \tag{15a}$$

$$\frac{1}{h}R_3^{(k)}-\frac{32I_3^{(k)}}{5n^{(k)2}h^3}+\frac{n^{(k)2}h}{140}(T_3^{(k-1)}-T_3^{(k)})$$

$$=\frac{11}{350}h^2n^{(k)3}c_{33}^{(k)}\xi_3+\frac{11}{1050}h^2n^{(k)3}\left[\tilde{\Psi}+(-1)^k\frac{2}{n^{(k)}h}\tilde{S}+3hn_0^{(k)}\tilde{\xi}+\frac{3h^2}{2}\left(5n_0^{(k)2}-\frac{1}{4}\right)\tilde{\Phi}\right], \tag{15b}$$

$$Q_3^{(k)}-\frac{14J_3^{(k)}}{(n^{(k)}h)^2}+\frac{n^{(k)}h}{12}(T_3^{(k-1)}+T_3^{(k)}) = -\frac{3h^3}{40}n^{(k)3}[\tilde{\xi}+5hn_0^{(k)}\tilde{\Phi}], \tag{15c}$$

$$\frac{1}{h}R_3^{(k)}-\frac{15I_3^{(k)}}{2n^{(k)2}h^3}+\frac{n^{(k)2}h}{96}(T_3^{(k-1)}-T_3^{(k)}) = -\frac{11h^4}{2688}n^{(k)5}\tilde{\Phi}, \tag{15d}$$

$$\frac{-11}{12}\left[\frac{Q_3^{(k)}}{c_{33}^{(k)}}+\frac{Q_3^{(k+1)}}{c_{33}^{(k+1)}}\right]+\frac{15}{2h}\left[\frac{R_3^{(k)}}{n^{(k)}c_{33}^{(k)}}-\frac{R_3^{(k+1)}}{n^{(k+1)}c_{33}^{(k+1)}}\right]$$

$$+\frac{55}{3h^2}\left[\frac{J_3^{(k)}}{n^{(k)2}c_{33}^{(k)}}+\frac{J_3^{(k+1)}}{n^{(k+1)2}c_{33}^{(k+1)}}\right]-\frac{70}{h^3}\left[\frac{I_3^{(k)}}{n^{(k)3}c_{33}^{(k)}}-\frac{I_3^{(k+1)}}{n^{(k+1)3}c_{33}^{(k+1)}}\right]$$

$$=\frac{h}{18}\left[\frac{n^{(k)}}{c_{33}^{(k)}}T_3^{(k-1)}+10\left(\frac{n^{(k)}}{c_{33}^{(k)}}+\frac{n^{(k+1)}}{c_{33}^{(k+1)}}\right)T_3^{(k)}+\frac{n^{(k+1)}}{c_{33}^{(k+1)}}T_3^{(k+1)}\right]. \tag{15e}$$

In eqns (14a–c) and (15a–d), k ranges from 1 to N, while in eqns (14d) and (15e), k ranges from 1 to $(N-1)$. Also, no summation on α is implied in eqn (14) and

$$\tilde{c}_\alpha^{(k)} \equiv \delta_{\alpha 1}c_{55}^{(k)}+\delta_{\alpha 2}c_{44}^{(k)}, \quad n_0^{(k)}=\frac{x_{30}^{(k)}}{h} \tag{16}$$

$$\begin{bmatrix}\tilde{U}\\ \tilde{\Psi}\\ \tilde{S}\\ \tilde{\xi}\\ \tilde{\Phi}\end{bmatrix}=\begin{bmatrix}U_{1,1} & U_{2,2}\\ \Psi_{1,1} & \Psi_{2,2}\\ S_{1,1} & S_{2,2}\\ \xi_{1,1} & \xi_{2,2}\\ \Phi_{1,1} & \Phi_{2,2}\end{bmatrix}\begin{bmatrix}c_{13}\\ c_{23}\end{bmatrix}^{(k)}. \tag{17}$$

By solving eqns (14) and (15), $Q_i^{(k)}, R_i^{(k)}, J_i^{(k)}, i_3^{(k)}$ and $T_i^{(k)}$ are obtained in terms of U_i, Ψ_i, S_i, ξ_i and Φ_α and their derivatives. As a result, the quantities N_{3i}, M_{3i}, K_{3i}, Z_{3i}, L_{3i} of eqn (13b) can be determined as functions of these displacement variables. Such expressions will automatically include the appropriate shear correction factors by virtue of the Reissner mixed variational principle.

The equilibrium equations (12) are supplemented with the following suitable boundary conditions:

$$\begin{aligned}&\text{specify } U_i \quad \text{or} \quad N_{\alpha i}\nu_\alpha\\ &\text{specify } \Psi_i \quad \text{or} \quad M_{\alpha i}\nu_\alpha\\ &\text{specify } S_i \quad \text{or} \quad Z_{\alpha i}\nu_\alpha\\ &\text{specify } \xi_i \quad \text{or} \quad L_{\alpha i}\nu_\alpha\\ &\text{specify } \Phi_\alpha \quad \text{or} \quad P_{\beta\alpha}\nu_\beta.\end{aligned} \tag{18}$$

The remaining constitutive equations for $N_{\alpha\beta}$, $M_{\alpha\beta}$, $Z_{\alpha\beta}$, $L_{\alpha\beta}$ and $P_{\alpha\beta}$ are obtained by substituting eqns (1a), (6) and (8b) into eqn (13a) to yield

$$\begin{bmatrix} \frac{1}{h}\underset{\sim}{N} \\ \frac{1}{h^2}\underset{\sim}{M} \\ \frac{1}{h}\underset{\sim}{Z} \\ \frac{1}{h^3}\underset{\sim}{L} \\ \frac{1}{h^4}\underset{\sim}{P} \end{bmatrix} = \begin{bmatrix} [N_U] & [N_\Psi] & 0 & [N_\xi] & [N_\Phi] \\ & [M_\Psi] & [M_S] & [M_\xi] & [M_\Phi] \\ & & \frac{1}{3}[N_U] & [Z_\xi] & [Z_\Phi] \\ \text{Symmetric} & & & [L_\xi] & [L_\Phi] \\ & & & & [P_\Phi] \end{bmatrix} \begin{bmatrix} \underset{\sim}{U} \\ \mathrm{h}\underset{\sim}{\Psi} \\ \underset{\sim}{S} \\ h^2\underset{\sim}{\xi} \\ \mathrm{h}^3\underset{\sim}{\Phi} \end{bmatrix}$$

$$+ \sum_{k=1}^{N} [C]^{(k)} \begin{bmatrix} \underset{\sim}{V}^N \\ \underset{\sim}{V}^M \\ \underset{\sim}{V}^Z \\ \underset{\sim}{V}^L \\ \underset{\sim}{V}^P \end{bmatrix} \begin{bmatrix} Q_3 \\ \frac{1}{h}R_3 \\ \frac{1}{h^2}J_3 \\ \frac{1}{h^3}I_3 \end{bmatrix}^{(k)}, \tag{19}$$

where $\underset{\sim}{N} = [N_{11}, N_{22}, N_{12}]^T$, $\underset{\sim}{U} = [U_{1,1}, U_{2,2}, U_{1,2}+U_{2,1}]^T$ with same expressions for $\underset{\sim}{M}$, $\underset{\sim}{\Psi} \ldots \underset{\sim}{P}$, $\underset{\sim}{\Phi}$, $[N_U] \ldots [P_\Phi]$ are 3×3 matrices, $[C]^{(k)}$ is a 15×5 matrix and $\underset{\sim}{V}^N \ldots \underset{\sim}{V}^P$ are 1×4 vectors.

3. BENDING OF RECTANGULAR LAMINATED PLATES

The proposed theory can be used to solve the bending problem of rectangular plates for which two opposite edges are simply supported. The other two edges can each have arbitrary boundary conditions. Here we assume that the edges parallel to the x_2-axis are simply supported, and the origin of the coordinate system is taken as shown in Fig. 2. The simply supported boundary conditions can be satisfied by trigonometric functions in x_1. The resulting ordinary differential equations in x_2 can be solved using the state–space concept.

The prescribed boundary conditions on the top and bottom surfaces of the plate are

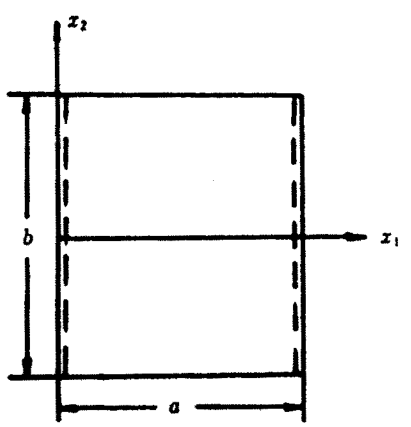

Fig. 2. Geometry and coordinate system of rectangular plate.

$$T_1^+ = T_2^+ = 0, \quad T_3^+ = q \quad \text{on}\, x_3 = \frac{h}{2} \tag{20a}$$

$$T_1^- = T_2^- = T_3^- = 0 \quad \text{on}\, x_3 = -\frac{h}{2}. \tag{20b}$$

The following representations of the displacements and loading are assumed:

$$\begin{bmatrix} U_1 \\ \Psi_1 \\ S_1 \\ \xi_1 \\ \Phi_1 \end{bmatrix} = \sum_{m=1}^{\infty} \begin{bmatrix} h\hat{U}_{1m}(x_2) \\ \hat{\Psi}_{1m}(x_2) \\ h\hat{S}_{1m}(x_2) \\ \frac{1}{h}\hat{\xi}_{1m}(x_2) \\ \frac{1}{h^2}\hat{\Phi}_{1m}(x_2) \end{bmatrix} \cos\alpha x_1 \quad \begin{bmatrix} U_2 \\ \Psi_2 \\ S_2 \\ \xi_2 \\ \Phi_2 \end{bmatrix} = \sum_{m=1}^{\infty} \begin{bmatrix} h\hat{U}_{2m}(x_2) \\ \hat{\Psi}_{2m}(x_2) \\ h\hat{S}_{2m}(x_2) \\ \frac{1}{h}\hat{\xi}_{2m}(x_2) \\ \frac{1}{h^2}\hat{\Phi}_{2m}(x_2) \end{bmatrix} \sin\alpha x_1$$

$$\begin{bmatrix} U_3 \\ \Psi_3 \\ S_3 \\ \xi_3 \end{bmatrix} = \sum_{m=1}^{\infty} \begin{bmatrix} h\hat{U}_{3m}(x_2) \\ \hat{\Psi}_{3m}(x_2) \\ h\hat{S}_{3m}(x_2) \\ \frac{1}{h}\hat{\xi}_{3m}(x_2) \end{bmatrix} \sin\alpha x_1 \tag{21}$$

$$q = \sum_{m=1}^{\infty} Q_m(x_2) \sin\alpha x_1, \tag{22}$$

where $\alpha = m\pi/a$ and $\hat{U}_{1m} \ldots \hat{\xi}_{3m}$ and Q_m denote amplitudes of $U_1 \ldots \xi_3$ and q, respectively. It is easily proven that eqn (21) can satisfy boundary condition of simply supported on $x_1 = 0, a$ i.e. at $x_1 = 0, a$

$$U_3 = \Psi_3 = S_3 = \xi_3 = 0 \quad \text{and} \quad N_{11} = M_{11} = Z_{11} = L_{11} = P_{11} = 0. \tag{23}$$

Then inserting eqn (21) into constitutive equations and these with eqn (22) in turn into the equilibrium eqn (12), yields a system of fourteen ordinary differential equations in the x_2-coordinate, which can be reduced to a single matrix differential equation using the state–space concept (Franklin, 1986)

$$X' = AX + B. \tag{24}$$

This can be done by introducing the variables

$$X = [\hat{U}'_{1m}\hat{U}_{1m}\hat{\Psi}'_{1m}\hat{\Psi}_{1m}\hat{S}'_{1m}\hat{S}_{1m}\hat{\xi}'_{1m}\hat{\xi}_{1m}\hat{\Phi}'_{1m}\hat{\Phi}_{1m}\hat{U}'_{2m}\hat{U}_{2m}\ldots\hat{\Phi}'_{2m}\hat{\Phi}_{2m}\hat{U}'_{3m}\hat{U}_{3m}\ldots\hat{\xi}'_{3m}\hat{\xi}_{3m}]^T,$$

where A is a 28×28 matrix which depends on the volume fractions $n^{(k)}$ and elastic constants $\bar{c}_{ij}$ and B is a 1×28 vector which depends on Q_m.

The solution of eqn (24) is given by

$$X = e^{Ax_2}K + e^{Ax_2}\int_{-b/2}^{x_2} e^{-A\eta}B\,d\eta, \tag{25}$$

where K is a 1×28 constant vector to be determined from the boundary conditions, while e^{Ax_2} is defined by

$$e^{Ax_2} = [L] \begin{bmatrix} e^{\lambda_1 x_2} & & 0 \\ & e^{\lambda_2 x_2} & \\ & & \ddots & \\ 0 & & & e^{\lambda_{28} x_2} \end{bmatrix} [L]^{-1}, \tag{26}$$

where $[L]$ is the matrix of eigenvectors, $\lambda_i (i = 1,2\ldots 28)$ denote the distinct eigenvalues associated with the matrix A and $[L]^{-1}$ is the inverse of the matrix $[L]$.

The following boundary conditions are used on the remaining two edges (i.e. the edges parallel to the x_1-axis) at $x_2 = \pm b/2$:

simply supported

$$\begin{aligned} U_1 &= \Psi_1 = S_1 = \xi_1 = \Phi_1 = 0 \\ U_3 &= \Psi_3 = S_3 = \xi_3 = 0 \\ N_{22} &= M_{22} = Z_{22} = L_{22} = P_{22} = 0; \end{aligned} \tag{27a}$$

clamped

$$\begin{aligned} U_1 &= \Psi_1 = S_1 = \xi_1 = \Phi_1 = 0 \\ U_2 &= \Psi_2 = S_2 = \xi_2 = \Phi_2 = 0 \\ U_3 &= \Psi_3 = S_3 = \xi_3 = 0; \end{aligned} \tag{27b}$$

free

$$\begin{aligned} N_{12} &= N_{22} = N_{23} = 0 \\ M_{12} &= M_{22} = M_{23} = 0 \\ Z_{12} &= Z_{22} = Z_{23} = 0 \\ L_{12} &= L_{22} = L_{23} = 0 \\ P_{12} &= P_{22} = 0. \end{aligned} \tag{27c}$$

4. NUMERICAL RESULTS AND DISCUSSION

The following numerical examples are presented:

(a) Numerical results are presented for orthotropic and symmetric cross-ply (0°/90°/0°) plate with same thickness layer subject to three types of loads; uniformly distributed load (q_0), triangular distributed load ($2q_0$) and concentrated load p, as shown in Fig. 3.

The following dimensionless orthotropic material properties are used:

$$\frac{E_1}{E_0} = 20.83 \qquad \frac{E_2}{E_0} = 10.94 \qquad E_3 = E_2$$

$$\frac{G_{12}}{E_0} = 6.10 \qquad \frac{G_{13}}{E_0} = 3.71 \qquad \frac{G_{23}}{E_0} = 6.19$$

$$\nu_{12} = \nu_{13} = \nu_{23} = 0.44 \qquad E_0 = 1 \times 10^6 \text{ psi}.$$

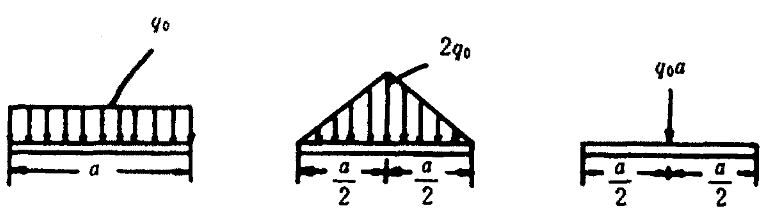

Fig. 3. Various types of transverse loads.

All results are compared with Khdeir's solution (Khdeir *et al.*, 1987) as shown in Tables 1–3. Tables 1–2 contain center deflections $\bar{u}_3$ for orthotropic and symmetric cross-ply (0°/90°/0°) plates, while Table 3 contains non-dimensionalized axial stresses $\bar{\sigma}_{11}$ for cross-ply (0°/90°/0°) plate.

The following notation has been used throughout the tables:

SS simply supported at $x_2 = -b/2$ and at $x_2 = b/2$;
CC clamped at $x_2 = -b/2$ and at $x_2 = b/2$;
FF free at $x_2 = -b/2$ and at $x_2 = b/2$;
SC simply supported at $x_2 = -b/2$ and clamped at $x_2 = b/2$;
SF simply supported at $x_2 = -b/2$ and free at $x_2 = b/2$;
CF clamped at $x_2 = -b/2$ and free at $x_2 = b/2$;
UN uniformly distributed load;
TR triangular distributed load;
PL point load at the center of the plate.

Figure 4 shows the thickness variations of axial stresses $\bar{\sigma}_{11}$ of (0°/90°/0°) laminated plate for various boundary conditions.

(b) Center deflections and stresses for cross-ply plates under sinusoidal transverse loading

$$\left(\text{i.e.}\quad q = q_0 \sin\frac{\pi}{a}x_1 \cos\frac{\pi}{b}x_2\right)$$

are calculated. The plates are simply supported at four edges. The numerical results are compared with exact elastic solution obtained by Pagano (1970).

The following material properties are used:

$$\frac{E_1}{E_0} = 25 \qquad \frac{E_2}{E_0} = 1.0 \qquad E_3 = E_2$$

$$\frac{G_{12}}{E_0} = 0.5 \qquad G_{13} = G_{12} \qquad \frac{G_{23}}{E_0} = 0.2$$

$$\nu_{12} = \nu_{13} = \nu_{23} = 0.25 \qquad E_0 = 1 \times 10^6 \text{ psi.}$$

We follow Pagano's non-dimensionalization and write the center deflection and stresses in the form:

$$\bar{u}_3 = \frac{100E_0h^3}{q_0a^4}u_3\left(\frac{a}{2},0,0\right)$$

$$\bar{\sigma}_{11} = \frac{h^2}{q_0a^2}\sigma_{11}\left(\frac{a}{2},0,x_3\right), \qquad \bar{\sigma}_{22} = \frac{h^2}{q_0a^2}\sigma_{22}\left(\frac{a}{2},0,x_3\right)$$

$$\bar{\tau}_{12} = \frac{h^2}{q_0a^2}\tau_{12}\left(0,-\frac{b}{2},x_3\right), \quad \bar{\tau}_{23} = \frac{h}{q_0a}\tau_{23}\left(\frac{a}{2},-\frac{b}{2},x_3\right)$$

$$\bar{\tau}_{31} = \frac{h}{q_0a}\tau_{31}(0,0,x_3), \qquad \bar{\sigma}_{33} = \frac{h}{q_0a}\sigma_{33}\left(\frac{a}{2},0,x_3\right).$$

Also

$$\bar{x}_3 = x_3/h \quad S = a/h.$$

Tables 4–6 show the center deflections and in-plane stresses and transverse shear stresses of the various side-to-thickness ratios for cross-ply rectangular plate (0°/90°/0°,

Table 1. Center deflections $\bar{u}_3$ of orthotropic plates

a/b	h/a	Loading	SS Khdeir's solution	SS Present solution	CC Khdeir's solution	CC Present solution	FF Khdeir's solution	FF Present solution	SC Khdeir's solution	SC Present solution	SF Khdeir's solution	SF Present solution	CF Khdeir's solution	CF Present solution
3	0.2	UN	6.29	5.632	3.19	3.183	224.4	221.015	4.38	4.137	50.14	49.025	17.055	17.064
		TR	9.60	8.537	5.29	5.198	289.2	284.581	6.98	6.509	66.30	64.607	24.49	23.688
		PL	20.95	18.564	14.46	13.131	371.4	363.787	17.09	15.071	93.46	89.541	40.99	38.443
	0.14	UN	14.23	13.512	5.89	5.881	593.1	590.463	8.74	8.498	124.72	121.714	40.13	39.771
		TR	21.15	19.874	9.71	9.592	761.9	756.991	13.71	13.017	163.50	160.533	55.13	53.976
		PL	42.38	39.931	25.71	24.277	966.8	958.142	31.83	29.976	222.8	199.124	87.19	84.818
4	0.2	UN	2.72	2.293	1.53	1.529	226.3	222.999	2.03	1.871	34.64	33.915	8.07	7.873
		TR	4.47	3.738	2.68	2.635	291.6	287.178	3.44	3.121	46.13	44.991	11.91	11.534
		PL	12.38	10.033	9.10	8.002	374.6	367.125	10.53	8.834	67.12	63.957	23.69	21.839
	0.14	UN	5.70	5.211	2.66	2.657	599.1	596.741	3.76	3.591	83.60	81.914	17.53	17.079
		TR	9.14	8.227	4.66	4.432	769.6	764.697	6.32	5.827	110.34	108.612	25.36	24.302
		PL	23.36	21.003	15.59	14.417	976.6	967.934	18.66	17.144	154.51	151.751	47.13	45.291
5	0.2	UN	1.46	1.148	0.88	0.883	227.1	223.913	1.14	1.014	25.97	25.465	4.29	4.213
		TR	2.52	1.959	1.59	1.552	292.8	288.391	2.01	1.748	34.76	33.936	6.71	6.512
		PL	8.39	6.204	6.32	5.327	376.1	368.745	7.27	5.721	51.86	49.151	15.77	14.267
	0.14	UN	2.85	2.494	1.49	1.492	601.8	598.976	2.03	1.911	60.68	58.196	8.87	8.403
		TR	4.84	4.014	2.70	2.613	773.2	767.899	3.56	3.071	80.51	78.877	13.58	12.978
		PL	15.15	13.455	10.69	9.884	981.3	972.176	12.57	11.224	115.45	113.006	29.86	28.035

$\bar{u}_3 = [u_3(a/2, 0, 0)/q_0]E_0, \quad a = 200\text{in}.$

Table 2. Center deflections $\bar{u}_3$ of cross-ply 0°/90°/0° laminates

			SS		CC		FF		SC		SF		CF	
a/b	h/a	Loading	Khdeir's solution	Present solution	Khdeir's solution	Present solution	Khdeir's solution	Present solution	Khdeir's solution	Present solution	Khdeir's solution	Present solution	Khdeir's solution	Present solution
3	0.2	UN	6.85	6.376	3.86	3.861	215.9	214.727	5.10	4.905	47.67	47.296	18.86	18.591
		TR	10.23	9.491	6.18	6.123	277.7	276.120	7.87	7.532	62.82	62.211	25.97	25.529
		PL	20.61	18.460	14.92	13.83	354.5	351.160	17.34	15.813	87.27	85.245	47.32	43.621
	0.14	UN	14.88	14.421	6.90	6.897	585.5	585.5	9.81	9.674	121.06	120.502	41.87	41.227
		TR	21.80	20.960	11.08	11.026	751.3	748.052	15.05	14.782	158.35	157.397	57.09	56.079
		PL	41.18	39.867	26.33	25.818	949.4	944.936	31.99	31.053	213.4	211.454	87.32	85.433
4	0.2	UN	3.12	2.811	1.87	1.872	217.8	216.723	2.43	2.274	32.34	32.198	9.03	8.902
		TR	4.99	4.409	3.19	3.135	280.2	278.713	4.00	3.709	42.98	42.663	13.06	12.813
		PL	12.47	10.871	9.48	8.485	357.6	354.452	10.85	9.375	61.72	60.065	24.09	22.678
	0.14	UN	6.23	5.927	3.21	3.181	591.4	589.061	4.38	4.212	80.07	79.785	18.88	18.544
		TR	9.78	9.159	5.47	5.431	758.9	755.841	7.18	6.912	105.47	105.008	26.99	26.453
		PL	23.01	20.813	16.16	15.832	959.1	945.404	19.00	18.368	146.04	144.705	47.58	46.573
5	0.2	UN	1.73	1.545	1.08	1.074	218.7	217.661	1.38	1.245	23.78	23.761	4.95	4.887
		TR	2.91	2.514	1.90	1.852	281.3	279.948	2.37	2.112	31.80	31.666	7.53	7.385
		PL	8.64	7.109	6.65	5.962	359.1	356.743	7.59	6.949	46.90	45.544	16.11	14.862
		UN	3.23	2.963	1.82	1.802	594.2	591.939	2.41	2.316	57.29	57.232	9.84	9.682
		TR	5.36	4.859	3.21	3.196	762.6	759.587	4.13	3.915	75.91	75.726	14.81	14.527
		PL	15.19	13.964	11.21	11.014	963.8	959.284	12.97	12.415	107.72	106.813	30.33	29.535

$\bar{u}_3 = [u_3(a/2, 0, 0)/q_0]E_0$, $a = 200$in.

Table 3. Axial center stresses $\bar{\sigma}_{11}$ of cross-ply 0°/90°/0° laminates

			SS		CC		FF		SC		SF		CF	
a/b	h/a	Loading	Khdeir's solution	Present solution	Khdeir's solution	Present solution	Khdeir's solution	Present solution	Khdeir's solution	Present solution	Khdeir's solution	Present solution	Khdeir's solution	Present solution
3	0.2	UN	1.082	0.973	0.482	0.480	19.66	19.082	0.729	0.691	4.272	4.064	1.098	1.020
		TR	2.115	1.808	1.262	1.134	26.51	25.554	1.619	1.426	6.513	6.064	2.402	2.148
		PL	11.160	10.911	9.792	9.735	43.66	42.344	10.386	10.036	17.708	17.112	12.419	12.143
	0.14	UN	2.106	1.982	0.794	0.729	39.54	38.911	1.262	1.209	8.519	8.003	1.622	1.541
		TR	3.983	3.623	2.077	1.934	53.01	51.648	2.784	2.479	12.766	11.879	3.799	3.481
		PL	17.928	17.001	14.814	14.211	82.76	80.043	16.049	15.133	30.78	28.333	19.17	18.327
4	0.2	UN	0.620	0.582	0.289	0.274	19.67	19.091	0.439	0.425	2.602	2.432	0.263	0.238
		TR	1.305	1.123	0.802	0.739	26.52	25.569	1.032	0.912	4.213	3.843	1.138	0.982
		PL	9.072	8.912	8.082	7.974	43.74	42.827	8.546	8.213	14.181	13.792	10.054	9.862
	0.14	UN	1.156	1.092	0.455	0.423	39.62	38.925	0.725	0.681	5.278	5.081	0.185	0.385
		TR	2.350	2.119	1.260	1.147	53.13	51.992	1.695	1.544	8.319	8.077	1.586	1.478
		PL	13.949	13.231	11.835	11.521	83.03	80.565	12.735	12.136	24.05	23.588	14.967	14.001
5	0.2	UN	0.427	0.404	0.215	0.211	19.66	19.098	0.321	0.315	1.673	1.519	0.027	0.024
		TR	0.927	0.803	0.588	0.533	26.51	25.573	0.756	0.697	2.879	2.552	0.662	0.612
		PL	7.725	7.589	6.890	6.571	43.75	43.142	7.298	6.943	11.859	11.512	8.680	8.328
	0.14	UN	0.759	0.720	0.321	0.306	39.63	38.952	0.507	0.481	3.492	3.315	−0.175	0.047
		TR	1.593	1.369	0.883	0.791	53.16	52.024	1.190	1.067	5.773	5.485	0.808	0.723
		PL	11.476	11.029	9.863	9.583	83.14	80.889	10.592	10.029	19.74	18.502	12.645	11.938

$\bar{\sigma}_{11} = \sigma_{11}(a/2, 0, h/2)/q_0E_0$, $a = 200$in.

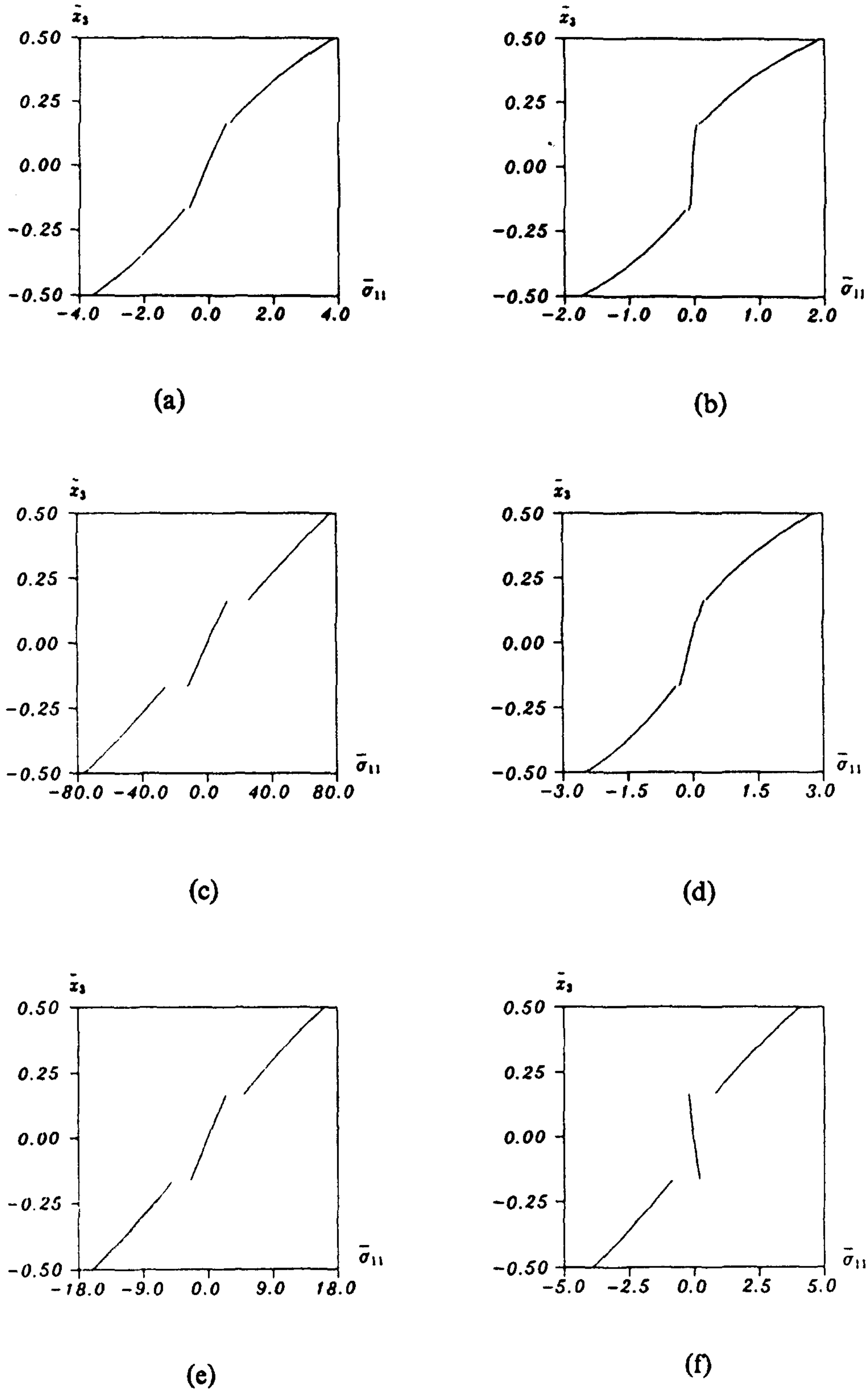

Fig. 4. Thickness variations of $\bar{\sigma}_{11}$ for various boundary conditions ($a/b = 3$, $a/h = 5$, UN): (a) – SSSS; (b) – SSCC; (c) – SSFF; (d) – SSSC; (e) – SSSF; (f) – SSCF.

$b/a = 3$) and square plates [0°/90°/90°/0°, 0°/90°…0° (9 layers)] with the same thickness layer, respectively.

For the side-to-thickness ratio $S = 0$, the thickness variations of in-plane and transverse shear and normal stresses for 0°/90°/0° laminated plate are shown in Fig. 5

Table 4. Center deflections and stresses of 0°/90°/0° laminates

S	Theory	$\bar{u}_3$	$\bar{\sigma}_{11}\left(\frac{h}{2}\right)$	$\bar{\sigma}_{22}\left(\frac{h}{6}\right)$	$\bar{\sigma}_{23}(0)$	$\bar{\sigma}_{31}(0)$	$\bar{\sigma}_{12}\left(\frac{h}{2}\right)$
	Pagano's solution	2.82	1.14	0.109	0.0334	0.351	−0.0269
4	Present solution	2.8234	1.1242	0.108	0.03276	0.3558	−0.02748
	Pagano's solution	0.919	0.726	0.0418	0.0152	0.420	−0.0120
10	Present solution	0.9148	0.7193	0.04152	0.01591	0.4143	−0.01204
	Pagano's solution	0.610	0.650	0.0299	0.0119	0.434	−0.0093
4	Present solution	0.6047	0.6439	0.02921	0.01282	0.4410	−0.0092
	Pagano's solution	0.508	0.624	0.0253	0.0108	0.439	−0.0083
100	Present solution	0.5029	0.6185	0.02507	0.0118	0.4459	−0.00824

From Tables 1–3, it is shown that the center deflections and stresses are slightly smaller than those obtained by Khdeir. This is because the present theory includes the effect of transverse normal strain (ε_{33}) and stress (σ_{33}). While this theory can satisfy the continuity condition of transverse shear stresses at the interfaces, this is not true for Khdeir's theory.

From Tables 4–6, close agreement for the center deflections and stresses of the present theory and the exact solution obtained by Pagano are observed for different side-to-thickness ratio and lamination schemes, which proves that the displacement field and trial transverse and normal stresses field of the present theory are appropriate and reasonable.

In the present work, transverse shear stresses τ_{31} and τ_{32} and transverse normal stress τ_{33} are obtained from eqn (8), which satisfy the top and bottom surfaces boundary

Table 5. Center deflections and stresses of 0°/90°/90°/0° laminates

S	Theory	$\bar{u}_3$	$\bar{\sigma}_{11}\left(\frac{h}{2}\right)$	$\bar{\sigma}_{22}\left(\frac{h}{4}\right)$	$\bar{\sigma}_{23}(0)$	$\bar{\sigma}_{31}(0)$	$\bar{\sigma}_{12}\left(\frac{h}{2}\right)$
	Pagano's solution	1.954	0.720	0.663	0.292	0.291	−0.0467
4	Present solution	1.884	0.7364	0.5908	0.2343	0.2285	−0.04612
	Pagano's solution	0.743	0.559	0.401	0.196	0.301	−0.0275
10	Present solution	0.7097	0.5499	0.3813	0.1548	0.3085	−0.02678
	Pagano's solution	0.517	0.543	0.308	0.156	0.328	−0.0230
10	Present solution	0.4980	0.5315	0.2984	0.1245	0.3340	−0.02246
	Pagano's solution	0.4385	0.539	0.276	0.141	0.337	−0.0216
100	Present solution	0.4247	0.5267	0.2648	0.1123	0.3440	−0.02087

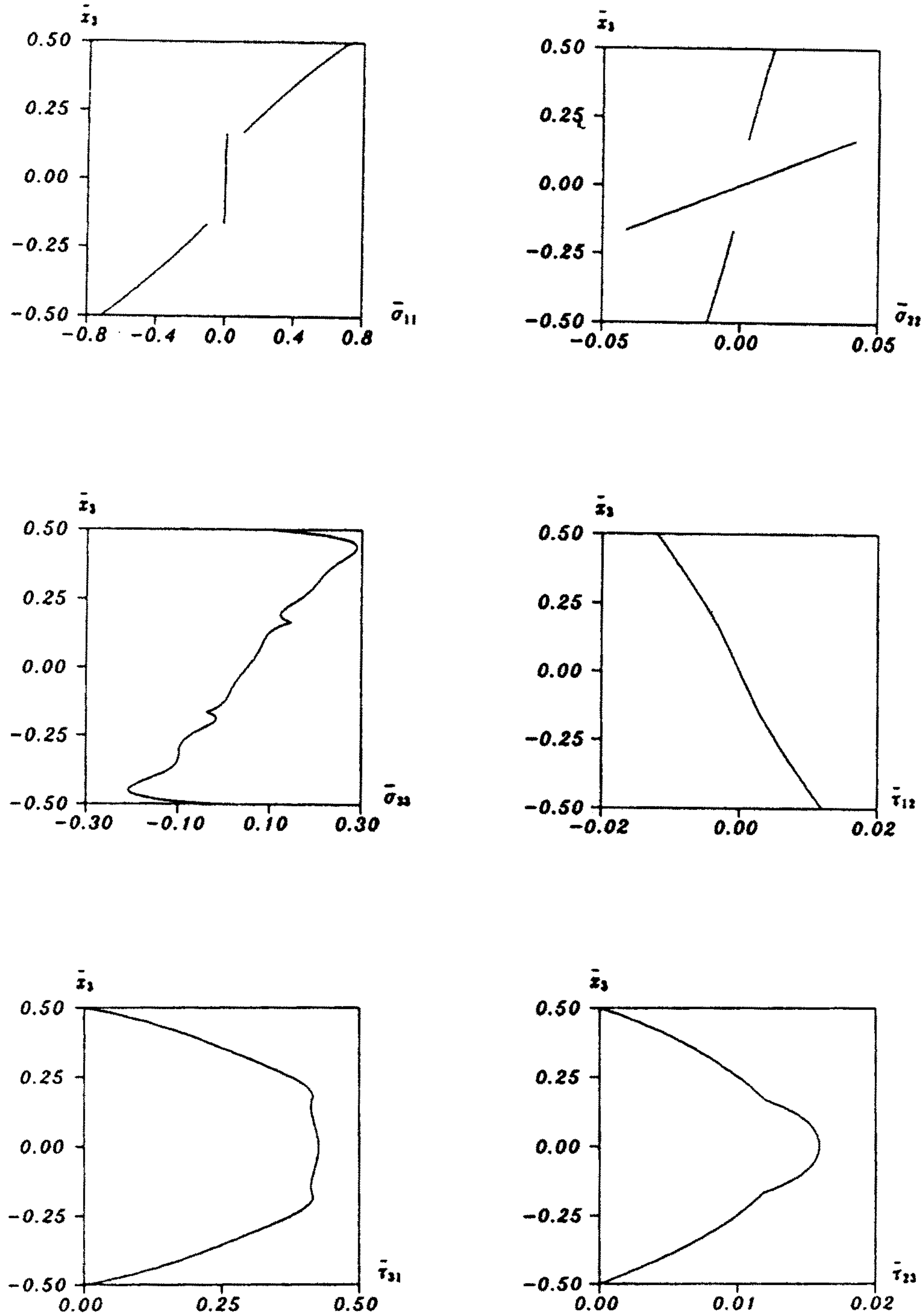

Fig. 5. Thickness variations of stresses.

conditions prescribed by eqn (20) (also see Fig. 5). Reddy (1984a) pointed out that the alternate procedure of computing the transverse stresses by integrating the equilibrium equations using the in-plane stresses found directly from the displacement solution yields more accurate results.

The first-order zig-zag theory proposed by Murakami (1986) is a particular case of this present theory. The author has applied it to the bending problem of rectangular plates. For 0°/90°/0° plate, better results were obtained, but for 0°/90°/0°/90° and 0°/90°/90°/0° plates, the first-order zig-zag theory deviates significantly from the exact solution. However, the present theory is still very good when compared with the exact solution. Obviously, the

Table 6. Center deflections and stresses of 0°/90°...0° (9 layers) laminates

S	Theory	$\bar{u}_3$	$\bar{\sigma}_{11}\left(\frac{h}{2}\right)$	$\bar{\sigma}_{22}\left(\frac{2h}{5}\right)$	$\bar{\sigma}_{23}(0)$	$\bar{\sigma}_{31}(0)$	$\bar{\sigma}_{12}\left(\frac{h}{2}\right)$
	Pagano's solution	1.7590	0.684		0.203	0.223	−0.0337
4	Present solution	1.7501	0.6620	0.02946	0.1990	0.2458	−0.03333
	Pagano's solution	0.6520	0.551		0.226	0.247	−0.0233
10	Present solution	0.6409	0.5341	0.02286	0.1878	0.2773	−0.02296

present theory is suitable for arbitrary laminated configurations, so it is the development of the first-order zig-zag theory.

5. CONCLUSION

An improved high-order shear deformation theory based upon Reissner's mixed variational principle in conjunction with the state–space concept is developed to determine the bending problems for rectangular laminated composite plate. Numerical results are presented for different edge conditions, aspect ratios, lamination schemes and loadings and are compared with Khdeir and Pagano's theories. The comparison indicates that the present theory accurately estimates in-plane responses, even for small side-to-thickness ratios and large layer laminates.

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